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Recent Results on N=2 Superconformal Algebras

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ABSTRACT

In the last six years remarkable developments have taken place concerning the representation theory of N=2 superconformal algebras. Here we present the highlights of such developments.

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1 Introduction to the N=2 Superconformal Algebras

1.1 The N=2 superconformal algebras

The N=2 superconformal algebras were discovered in the seventies independently by Ademollo et al. [1] and by Kac [2]. The first authors derived the algebras for physical purposes, in order to define supersymmetric strings, whereas Kac derived them for mathematical purposes along with his classification of Lie superalgebras. In modern notation these algebras read

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \tfrac{\mathtt{c}}{12}(m^3 - m)\delta_{m+n,0}\,, & [H_m, H_n] &= \tfrac{\mathtt{c}}{3}m\delta_{m+n,0}\,,\\
[L_m, G_r^{\pm}] &= \left(\tfrac{m}{2} - r\right) G_{m+r}^{\pm}\,, & [H_m, G_r^{\pm}] &= \pm G_{m+r}^{\pm}\,,\\
[L_m, H_n] &= -\,nH_{m+n} & &\\
\{G_r^-, G_s^+\} &= 2L_{r+s} - (r-s)H_{r+s} + \tfrac{\mathtt{c}}{3}(r^2 - \tfrac{1}{4})\delta_{r+s,0}\,. & &
\end{aligned} \tag{1.1}$$

The bosonic generators L_n and H_n correspond to the stress-energy tensor (Virasoro generators) and to the U(1) current, respectively. The fermionic generators $G_r^{\pm}$, with conformal weight 3/2, correspond to the two fermionic currents. Depending on the modings of the algebra generators $G_r^{\pm}$ and H_n one distinguishes three N=2 algebras: the Neveu-Schwarz, the Ramond and the twisted N=2 algebras. For the Neveu-Schwarz N=2 algebra the two fermionic generators $G_r^{\pm}$ are half-integer moded and the U(1) generators H_n are integer moded, i.e. $r \in \mathbf{Z} + 1/2,\ \ n \in \mathbf{Z}$. For the Ramond N=2 algebra $G_r^{\pm}$ and H_n are integer moded, i.e. $r \in \mathbf{Z},\ \ n \in \mathbf{Z}$ and for the twisted N=2 algebra G_r^+ is integer moded whereas G_s^- and H_n are half-integer moded, i.e. $r \in \mathbf{Z},\ \ s \in \mathbf{Z}{+}1/2,\ \ n \in \mathbf{Z}{+}1/2$.

The Neveu-Schwarz, the Ramond and the twisted N=2 superconformal algebras are not the whole story, however. In 1990 Dijkgraaf, Verlinde and Verlinde [3] presented the Topological N=2 superconformal algebra, which is the symmetry algebra of Topological Conformal Field Theory in two dimensions. This algebra can be obtained from the Neveu-Schwarz N=2 algebra by 'twisting' the stress-energy tensor by addding the derivative of the U(1) current, procedure known as 'topological twist' [4][5]. The Topological N=2 algebra reads

$$\begin{aligned}
[\mathcal{L}_m, \mathcal{L}_n] &= (m-n)\mathcal{L}_{m+n}\,, & [\mathcal{H}_m, \mathcal{H}_n] &= \tfrac{\mathtt{c}}{3}m\delta_{m+n,0}\,,\\
[\mathcal{L}_m, \mathcal{G}_n] &= (m-n)\mathcal{G}_{m+n}\,, & [\mathcal{H}_m, \mathcal{G}_n] &= \mathcal{G}_{m+n}\,,\\
[\mathcal{L}_m, \mathcal{Q}_n] &= -n\mathcal{Q}_{m+n}\,, & [\mathcal{H}_m, \mathcal{Q}_n] &= -\mathcal{Q}_{m+n}\,, \qquad m,\ n \in \mathbf{Z}\,.\\
[\mathcal{L}_m, \mathcal{H}_n] &= -n\mathcal{H}_{m+n} + \tfrac{\mathtt{c}}{6}(m^2+m)\delta_{m+n,0}\,, & &\\
\{\mathcal{G}_m, \mathcal{Q}_n\} &= 2\mathcal{L}_{m+n} - 2n\mathcal{H}_{m+n} + \tfrac{\mathtt{c}}{3}(m^2+m)\delta_{m+n,0}\,, & &
\end{aligned} \tag{1.2}$$

The fermionic generators $\mathcal{G}_n$ and $\mathcal{Q}_n$, with conformal weight 2 and 1, respectively, and the generators $\mathcal{H}_n$ are integer moded, i.e. $n \in \mathbf{Z}$. The existence of two fermionic zero modes for the Ramond and for the Topological N=2 algebras allows one to classify the states in the corresponding Verma modules into two sectors: the $(+)$ and $(-)$ sectors for the Ramond states, G_0^- and G_0^+ interpolating between them, and the $\mathcal{G}$ and $\mathcal{Q}$ sectors for the topological states, $\mathcal{Q}_0$ and $\mathcal{G}_0$ interpolating between them. The two sectors are not the complete description of the states, however, since there exist indecomposible states in the Verma modules which do not belong to any of the sectors [23][29][30][32]. In the case of the Ramond N=2 algebra the indecomposible states are called 'no-helicity' states and in the case of the Topological N=2 algebra they are called 'no-label' states.

1.2 Relations between the N=2 superconformal algebras

The Neveu-Schwarz and the Ramond N=2 algebras are connected through the spectral flows. Namely, the even ($\mathcal{U}_\theta$) and the odd ($\mathcal{A}_\theta$) spectral flows transform the Neveu-Schwarz and the Ramond N=2 algebras into each other for $\theta = \mathbf{Z} + 1/2$. However, they do not map highest weight (h.w.) vectors into h.w. vectors, but only in particular cases, and they deform the Verma modules very much, so that they are not isomorphic.

The Neveu-Schwarz and the Topological N=2 algebras are connected through the topological twists $T_W^{\pm}$. Under these only the topological h.w. vectors annihilated by $\mathcal{G}_0$ are transformed into h.w. vectors of the Neveu-Schwarz N=2 algebra and the Verma modules are deformed in a similar way as by the action of the spectral flows for $\theta = \pm 1/2$. Thus the corresponding Verma modules are not isomorphic. The topological twists $T_W^{\pm}$ consists of the modification of the stress-energy tensor by adding the derivative of the U(1) current: $T(z) \to T(z) \pm 1/2\ \partial H(z)$. As a result the conformal weights (spins) of the fermionic fields are modified by $\pm 1/2$, what automatically produces a shift of $\pm 1/2$ in the modings of these generators: the spin-3/2 fermionic generators G_r^+, G_r^-, with $r \in \mathbf{Z} + 1/2$, are transformed into the spin-2 and spin-1 fermionic generators $\mathcal{G}_n$ and $\mathcal{Q}_n$, respectively, with $n \in \mathbf{Z}$. In addition, $\mathcal{Q}(z)$ has the properties of a BRST-current, $\mathcal{Q}_0$ being the BRST charge. The topological nature of the resulting algebra manifests itself through the BRST-exactness of the stress-energy tensor: $\mathcal{L}_m = 1/2\{\mathcal{Q}_0, \mathcal{G}_m\}$. When this occurs, the correlators of the fields of the superconformal field theory do not depend on the two-dimensional metric, as is well known in the literature (see for example ref. [6]).

Finally, the Ramond and the Topological N=2 algebras are connected through the composition of the spectral flows with $\theta = \mathbf{Z} + 1/2$ and the topological twists. For $\theta = \pm 1/2$ one finds exact isomorphisms between the states of these algebras level by level [32]. The Verma modules of the Ramond and of the Topological N=2 algebras are therefore isomorphic.

2 Representation Theory of the N=2 Superconformal Algebras

2.1 Historical overview

Let us now briefly review the most important developments concerning the representation theory of the N=2 superconformal algebras. One can distinguish two periods of remarkable activity. In the first period, from 1985 until 1988, the determinant formulae for the Neveu-Schwarz, the Ramond and the twisted N=2 algebras were written down by various authors, unitarity of the representations was analyzed and some singular vectors (called simply null vectors) were computed [7][8][9][10][11][12][13]. Also some embedding diagrams were presented [12][14][15] and the (even) spectral flows interpolating between the Neveu-Schwarz and the Ramond N=2 algebras were written down [16].

During the period from 1989 until 1993 there was not much activity in the representation theory of N=2 algebras. However, two developments took place which were of crucial importance. On one side the chiral rings were discovered [17] and with them one realized the necessity to analyze the chiral representations of the N=2 algebras. On the other side the Topological N=2 algebra was written down [3] and its importance for string theory was also realized [18][19].

In the second period of remarkable activity, from 1994 until nowadays, there have been several important findings regarding the states in the Verma modules of the N=2 algebras: the discovery of two-dimensional singular spaces [25][31], the discovery of subsingular vectors [22][23][24][31] and the discovery of indecomposible states [23][32]. In addition an (almost) complete classification of embedding diagrams was presented for the Neveu-Schwarz N=2 algebra [26][27] and the determinant formulae for the Topological N=2 algebra were computed [32] as well as the determinant formulae for the chiral representations of the Topological, the Neveu-Schwarz and the Ramond N=2 algebras [24]. Moreover, the odd spectral flows were written down [20][21], which are believed to provide the complete set of automorphisms for the N=2 algebras. Furthermore, recently a powerful tool has been developed for the study of the representations of any Lie algebra or superalgebra, the so-called 'adapted ordering method'. This method has been applied so far sucessfully to the N=2 algebras and to the Ramond N=1 algebra. In what follows we will say a few words about all these new results and developments.

2.2 The odd spectral flow

The odd spectral flow $\mathcal{A}_\theta$, when acting on the states and generators of the Neveu-Schwarz and the Ramond N=2 algebras read [20][21]

$$
\begin{aligned}
\mathcal{A}_\theta\, L_m\, \mathcal{A}_\theta^{-1} &= L_m + \theta H_m + \tfrac{\mathsf{c}}{6}\theta^2\delta_{m,0}\,, \\
\mathcal{A}_\theta H_m\, \mathcal{A}_\theta^{-1} &= -H_m - \tfrac{\mathsf{c}}{3}\theta\delta_{m,0}\,, \\
\mathcal{A}_\theta\, G_r^+\, \mathcal{A}_\theta^{-1} &= G_{r-\theta}^-\,, \\
\mathcal{A}_\theta\, G_r^-\, \mathcal{A}_\theta^{-1} &= G_{r+\theta}^+\,,
\end{aligned}
\tag{2.1}
$$

with $\mathcal{A}_\theta^{-1} = \mathcal{A}_\theta$ (it is therefore an involution). $\mathcal{A}_\theta$ and the even (usual) spectral flow $\mathcal{U}_\theta$ [16] are quasi-mirror symmetric under $H_m \leftrightarrow -H_m,\ G_r^+ \leftrightarrow G_r^-,\ \theta \leftrightarrow -\theta$. However, $\mathcal{A}_\theta$ generates $\mathcal{U}_\theta$ and consequently it is the only fundamental spectral flow, as one can see in the composition rules

$$
\mathcal{U}_{\theta_2}\, \mathcal{U}_{\theta_1} = \mathcal{U}_{(\theta_2+\theta_1)}, \qquad \mathcal{A}_{\theta_2}\, \mathcal{A}_{\theta_1} = \mathcal{U}_{(\theta_2-\theta_1)}, \tag{2.2}
$$

$$
\mathcal{A}_{\theta_2}\, \mathcal{U}_{\theta_1} = \mathcal{A}_{(\theta_2-\theta_1)}, \qquad \mathcal{U}_{\theta_2}\, \mathcal{A}_{\theta_1} = \mathcal{A}_{(\theta_2+\theta_1)}. \tag{2.3}
$$

$\mathcal{A}_\theta$ is believed to provide the complete set of automorphisms of the N=2 algebras (for $\theta \in \mathbf{Z}$), whereas $\mathcal{U}_\theta$ provide only 'half' of them.

2.3 Indecomposible singular vectors

The indecomposible singular vectors of the Ramond N=2 algebra were overlooked until very recently (they do not exist for the Neveu-Schwarz N=2 algebra). In fact, they were discovered first for the Topological N=2 algebra in ref. [23], where they were called 'no-label' singular vectors. Shortly afterwards some examples were presented for the Ramond N=2 algebra [29] under the name 'no-helicity' singular vectors.

At first sight, just by inspecting the anticommutator of the fermionic zero modes, one realizes that indecomposible singular vectors are allowed to exist by the Topological (and Ramond) N=2 algebra. Namely, from $\{\mathcal{G}_0, \mathcal{Q}_0\} = 2\mathcal{L}_0$ one deduces that for non-zero conformal weight $\Delta \neq 0$ all the states can be decomposed into linear combinations of $\mathcal{G}_0$-closed states and $\mathcal{Q}_0$-closed states (i.e. states annihilated by $\mathcal{G}_0$ and states annihilated by $\mathcal{Q}_0$):

$$
|\chi\rangle = \frac{1}{2\Delta}(\mathcal{G}_0\mathcal{Q}_0|\chi\rangle + \mathcal{Q}_0\mathcal{G}_0|\chi\rangle)\,. \tag{2.4}
$$

For zero conformal weight $\Delta = 0$, however, there is not such a decomposition and the states can be annihilated either by one of the fermionic zero modes or by both or by none of them. The latter are the indecomposible 'no-label' states. In the case of the Ramond N=2 algebra, from the anticommutator $\{G_0^+, G_0^-\} = 2L_0 - \frac{\mathsf{c}}{12}$ one deduces that the 'no-helicity' indecomposible states are allowed for $\Delta = \frac{\mathsf{c}}{24}$.

Indecomposible singular vectors are also allowed by the analysis of maximal dimensions [30] (see later). They actually exist already at levels 1 and 2 [23][29] and recently it has been proved that they *must* necessarily exist [32]. The argument goes as follows. One analyzes two curves in the parameter space of singular vectors. Each curve corresponds to two different families of singular vectors. In some discrete intersection points, however, the four singular vectors reduce to three (two of them coincide). But the rank of the inner product matrix is upper semi-continuous and therefore another singular vector must exist. By analyzing the dimensions of the singular vectors involved one deduces finally that the new singular vector must be indecomposible.

2.4 New determinant formulae

The determinant formulae for the Topological N=2 algebra have been computed for generic (standard) Verma modules, for chiral Verma modules and for no-label Verma modules [24][32]. In addition, determinant formulae have been computed for the chiral Verma modules of the Neveu-Schwarz and of the Ramond N=2 algebras [24], as well as for the no-helicity Verma modules of the Ramond N=2 algebra (as a straightforward derivation of the results for the no-label Verma modules of the Topological N=2 algebra) [32].

The generic Verma modules of the Topological N=2 algebra are built on $\mathcal{G}_0$-closed and/or $\mathcal{Q}_0$-closed h.w. vectors (annihilated either by $\mathcal{G}_0$ or by $\mathcal{Q}_0$). For conformal weight $\Delta \neq 0$ there are two h.w. vectors at the bottom of the Verma modules (one is $\mathcal{G}_0$-closed and the other $\mathcal{Q}_0$-closed), giving rise to the two sectors: the $\mathcal{G}$-sector and the $\mathcal{Q}$-sector. For $\Delta = 0$, however, there is only one h.w. vector (plus one singular vector at level zero) at the bottom of the Verma modules.

The chiral Verma modules of the Topological N=2 algebra are built on chiral h.w. vectors annihilated by both $\mathcal{G}_0$ and $\mathcal{Q}_0$. Therefore they have only one h.w. vector at the bottom, which has zero conformal weight $\Delta = 0$. They are incomplete Verma modues that can be realized as quotient modules of a generic Verma module with $\Delta = 0$ divided by the submodule generated by its level zero singular vector. Similarly, the chiral Verma modules of the Ramond N=2 algebra are built on h.w. vectors annihilated by both G_0^+ and G_0^-, with $\Delta = \frac{\mathtt{c}}{24}$. The chiral (anti-chiral) Verma modules of the Neveu-Schwarz N=2 algebra, in turn, are built on chiral (anti-chiral) h.w. vectors annihilated by $G_{-1/2}^+$ ($G_{-1/2}^-$), satisfying $\Delta = \frac{h}{2}$ ($\Delta = \frac{-h}{2}$), where h is the U(1) charge.

Finally, no-label (no-helicity) Verma modules are built on no-label (no-helicity) h.w. vectors. At the bottom they consist of one h.w. vector plus three singular vectors at level zero obtained by the action of $\mathcal{G}_0$ and $\mathcal{Q}_0$ (G_0^+ and G_0^-) on the no-label (no-helicity) h.w. vector. No-label (no-helicity) Verma modules *appear as submodules* inside generic Verma modules, the bottom of these submodules consisting of four singular vectors consequently.

Apart from no-label (no-helicity) submodules, in generic Verma modules of the Topological (Ramond) N=2 algebra one can find another three types of submodules, taking

into account the size and the shape at the bottom of the submodule [32]. In chiral Verma modules, however, one finds only one kind of submodule [24][32].

2.5 Discovery of subsingular vectors

Subsingular vectors are null, but not h.w. vectors, that become singular (i.e. h.w. vectors) after the quotient of the Verma module by a submodule. That is, they become singular by setting one singular vector to zero. This implies that they are located *outside* that submodule since otherwise they would go away after the quotient.

Subsingular vectors for the Neveu-Schwarz, the Ramond and the Topological N=2 algebras were reported for the first time in 1996 - 1997 in refs. [22][23][24]. For the twisted N=2 algebra subsingular vectors were presented only recently in ref. [31]. The discovery of subsingular vectors for the N=2 algebras was as follows. From the chiral determinant formulae for the Neveu-Schwarz, the Ramond and the Topological N=2 algebras [24] one deduces the existence of singular vectors in the chiral Verma modules that are not singular in the complete (generic) Verma modules before the quotient that gives rise to the chiral Verma modules. These are therefore subsingular vectors in the generic Verma modules. Many explicit examples of subsingular vectors of this type (i.e. becoming singular in the chiral Verma modules) were presented for the Neveu-Schwarz, the Ramond and the Topological N=2 algebras [23][24]. For the twisted N=2 algebra subsingular vectors were found by analyzing the Verma modules using the 'adapted ordering method' [31] (see later).

2.6 Discovery of two-dimensional singular vector spaces

The two-dimensional singular vector spaces were discovered first for the Neveu-Schwarz N=2 algebra by Dörrzapf [25][26] in 1994. In particular, some conditions were found to guarantee the existence of two-dimensional spaces of *uncharged* singular vectors. These conditions can be written as the simultaneous vanishing of two functions: $\epsilon_r^+(h, \mathsf{c}) = \epsilon_r^-(h, \mathsf{c}) = 0$, where h is the U(1) charge of the h.w. vector of the Verma module. The corresponding uncharged singular vectors are located in Verma modules where two *charged* and one *uncharged* singular vectors intersect (at different levels) in such a way that the charged singular vectors are the primitive ones and the two-dimensional uncharged singular space is secondary of the two charged singular vectors. These two-dimensional spaces are spanned by a tangent space of vanishing surfaces corresponding to singular vectors. Once the 'general formula' for singular vectors vanish one gets the two-dimensional singular space in the tangent space.

An straightforward extension of these results to the Topological N=2 algebra was presented in ref. [23]. As a result four types of topological singular vectors were found for which two-dimensional spaces may exist. The extension of these results to the Ramond N=2 algebra is also straightforward since the corresponding Verma modules are isomor-

phic to the Verma modules of the Topological N=2 algebra. For the twisted N=2 algebra two-dimensional singular spaces were found using the adapted ordering method [31]. In this case, however, the two-dimensional singular spaces are of different nature than the ones corresponding to the other N=2 algebras: they are spanned by two *primitive* singular vectors instead of two secondary singular vectors.

2.7 N=2 Embedding diagrams

In 1995 Dörrzapf presented a complete classification of embedding diagrams for the Neveu-Schwarz N=2 algebra [26] (see also ref. [27]). He proved that the relative charge q of all singular vectors (not only the primitive ones) satisfy $|q| \leq 1$, correcting many earlier diagrams in the literature [12], and he presented many more diagrams than previously known [14][15]. These results have not been improved so far although we know that they must be improved because subsingular vectors were assumed not to exist, being discovered one year afterwards.

The embedding diagrams of the Neveu-Schwarz N=2 algebra can be carefully adapted to provide embedding diagrams for the Topological and for the Ramond N=2 algebras. One has to take into account, however, that many singular vectors of these algebras do not correspond to singular vectors of the Neveu-Schwarz N=2 algebra but to null descendants of singular vectors or even to subsingular vectors (for example, indecomposable no-label and no-helicity singular vectors of the Topological and of the Ramond N=2 algebras always correspond to subsingular vectors of the Neveu-Schwarz N=2 algebra [29]).

3 The Adapted Ordering Method

The ‘adapted ordering method’, developed in ref. [30], can be applied to most Lie algebras and superalgebras. It allows:

i) to determine *maximal dimensions* for a given type of singular vector space,

ii) to rule out the existence of certain types of singular vectors (with dimension zero),

iii) to identify all singular vectors by only a few coefficients,

iv) to spot subsingular vectors,

v) to obtain easily product expressions of singular vector operators in order to compute secondary singular vectors (or decide whether they vanish),

vi) to set the basis for constructing embedding diagrams.

The method originates (in rudimentary form) from a procedure developed by Kent for the analytically continued Virasoro algebra [33]. The analytical continuation is not necessary, however, for the adapted ordering method. The key idea of this method is to find a suitable ordering for the terms of the singular vectors, i.e. a criterion to decide which of two terms is the bigger one, for example between the terms at level 4 and

charge 1: $G^+_{-2}L_{-2}$ and $G^+_{-1}H_{-2}L_{-1}$. The ordering must be *adapted* to a subset of terms $\mathcal{C}^A_{l,q} \in \mathcal{C}_{l,q}$, where $\mathcal{C}_{l,q}$ is the set of all possible terms at level l with charge q (for the details see ref. [30]).

The crucial point of this method is the following. The complement of $\mathcal{C}^A_{l,q}$ is the ordering kernel $\mathcal{C}^K_{l,q} = \mathcal{C}_{l,q}/\mathcal{C}^A_{l,q}$ and its size puts a limit on the dimension of the corresponding singular vector space. Namely, if the ordering kernel $\mathcal{C}^K_{l,q}$ has n elements then there are at most n linearly independent singular vectors $\Psi_{l,q}$, at level l with charge q, in a given Verma module. In other words, the singular vectors $\Psi_{l,q}$ span a singular space that is at most n-dimensional. As a result, if $\mathcal{C}^K_{l,q} = \emptyset$ then there are no singular vectors of type $\Psi_{l,q}$ (the singular space has dimension zero). Therefore we need to find a suitable, clever ordering in order to obtain the smallest possible kernel. Furthermore, the coefficients with respect to the terms of the ordering kernel uniquely identify a singular vector. This implies that just a few (one, two,...) coefficients completely determine a singular vector no matter its size. As a consequence one can find easily product expressions for descendant singular vectors and set the basis to construct embedding diagrams.

The adapted ordering method has been applied to the Topological, to the Neveu-Schwarz and to the Ramond N=2 algebras in ref. [30] and to the twisted N=2 algebra in ref. [31]. The maximal dimensions of the existing types of singular vectors have been found to be one or two, with the exception of some types of singular vectors in 'no-label' and 'no-helicity' Verma modules for which the maximal dimension has been found to be three. For the Topological and the Ramond N=2 algebras the only existing types of singular vectors (primitive as well as secondary), distinguished by the relative charge q and the annihilation properties under the fermionic zero modes, resulted in: twenty types in generic Verma modules, with $|q| \leq 2$, nine types in no-label and no-helicity Verma modules, with $|q| \leq 2$, and four types in chiral Verma modules, with $|q| \leq 1$. These results had been conjectured previously in ref. [23]. For the Neveu-Schwarz N=2 algebra one obtained $|q| \leq 1$, in agreement with the results presented in ref. [25]. As we pointed out before, in the case of the twisted N=2 algebra the application of the adapted ordering method has lead to the discovery of subsingular vectors and two-dimensional singular spaces for this algebra.

4 Final Remarks

In spite of the progress made in the last six years, the representation theory of the N=2 superconformal algebras is not finished yet. It remains to classify the subsingular vectors and to complete the classification of embedding diagrams. Several of the techniques that have been used for the analysis of these algebras can be easily transferred to the analysis of other Lie algebras and superalgebras. This holds especially for the adapted ordering method, that has already been applied sucessfully to the Ramond N=1 superconformal algebra [28], leading as a result to the discovery of two-dimensional singular spaces and subsingular vectors, which do not exist for the Neveu-Schwarz N=1 superconformal

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